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Boundary Approximation Methods for Solving Elliptic Problems on Unbounded Domains

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Boundary approximation methods with partial solutions are presented for solving a complicated problem on an unbounded domain, with both a crack singularity and a corner singularity. Also an analysis of partial solutions near the singular points is provided. These methods are easy to apply, have good stability properties, and lead to highly accurate solutions. Hence, boundary approximation methods with partial solutions are recommended for the treatment of elliptic problems on unbounded domains provided that piecewise solution expansions, in particular, asymptotic solutions near the singularities and infinity, can be found. © 1990 Academic Press, Inc.

1. INTRODUCTION

Boundary approximation methods with partial solutions are presented in Li, Mathon, and Sermer [8] for solving elliptic equations on bounded domains. In this paper, we will apply these methods to solve a complicated problem in an

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unbounded domain, with both a crack and a corner singularity. The singularity of the solution at the corner is of the order

$$u = O(1) + O(\rho^2 \ln \rho) \qquad \text{as} \quad \rho \to 0 \tag{1.1}$$

and will be called *a mild singularity.*

In our application, the solution domain is divided into several subdomains. Different expansions of solutions are used in different subdomains. Assuming that the expansions used satisfy exactly the elliptic equations within subdomains, an approximate solution is then found by obtaining the expansion coefficients that satisfy, as best as possible in a least-squares sense, the exterior boundary conditions and the interior continuity conditions on the common boundary of subdomains. Since the partial solutions are chosen to satisfy the equation in the subdomains and since the solution procedure is performed along the interior and exterior boundaries, we call these methods *boundary approximation methods with partial solutions* (BAM) to distinguish it from the boundary element methods (BEM). Evidently, BAM are derived from the Trefftz method [12], where unified expansions of solutions have to be employed in the entire solution domain. In this paper we use "partial solutions" for the solutions satisfying exactly the *homogeneous* equations.

Since local expansions or asymptotic forms of solutions satisfy the differential equation, BAM can be employed to solve problems with both singularities and unbounded domains. Also, highly accurate approximations can be obtained by using relatively few expansion terms, thus requiring only a modest computational effort regarding both CPU time and storage. In addition, the approximate solutions obtained are piecewise analytical so that BAM are beneficial to the analysis of engineering problems. These all are advantages over finite element and finite difference methods.

However, in applying BAM, we have to choose appropriate piecewise expansions of the solutions. This is required even by other numerical methods, i.e., the generalized finite element analysis of Zielinski and Zienkiewicz [12]. Fortunately, the textbooks of partial differential equations (i.e., Tikhonov and Samarskii [10]) provide useful partial solutions for the most important equations arising in applications. But sometimes, an analysis is needed to find suitable solution expansions. Such an analysis is essential for problems with singularities because the asymptotic behavior of the solutions near the singular points or at infinity is often unknown or unclear.

The stability of numerical solutions obtained by using BAM is also important. It is shown in [8] that the stability can be improved if several subdomains are used. For BAM, both the accuracy and stability of numerical solutions obtained rely substantially on how we choose:

(1) Geometric shapes of subdomains and

(2) Piecewise partial solutions on the subdomains.

If we are able to deal with these two aspects properly, we can obtain very accurate solutions and very small condition numbers. Consequently, an effort is made to find appropriate kinds of singular solutions in this paper. An investigation of geometric shapes of subdomains will be pursued in Li and Mathon [7].

In fact, approaches similar to BAM may have been used in some engineering problems. A precise description of the methods, convergence proofs, and error estimates for numerical solutions are given in Li [5] and Li, Mathon, and Sermer [8]. It is worthwhile to point out that BAM and its theoretical analysis presented in [8] can be easily extended to unbounded domain problems for the equation (Babuska and Aziz [1] and Fufner [3]),

$$-\Delta u + cu = 0, \qquad 0 < \alpha < c < \beta, \tag{1.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and α and β are constants. Therefore, we will take the Debye–Huckel equation,

$$-\Delta u + u = 0, \tag{1.3}$$

which results from the theory of weak electrolytes, as a sample equation. Equation (1.3) is, in fact, also a typical elliptic equation. To solve Eq. (1.3) we shall describe a BAM and carry out numerical experiments.

In the next two sections, we will find local partial solutions on all subdomains, which will then form a basis for the approximate solutions used by the BAM. Numerical experiments and discussions are given in the last section.

2. Partial Solutions on Subdomains

Let Ω^* be the upper semi-plane, excluding the cut $(x=0) \cap (0 \leqslant y \leqslant 1)$. Now we consider the elliptic equation in Ω^*,

$$-\Delta u + u = 0, \qquad (x, y) \in \Omega^*,$$

and the Dirichlet conditions (see Fig. 1):

$$u = 1 \quad \text{for} \quad x = 0 \text{ and } 0 < y < 1; \qquad \text{and} \qquad u = 1 \quad \text{for} \quad y = 0.$$

In two dimensions (see Tikhonov and Samarskii [10]), the infinity condition of solutions for this problem is the property of boundedness: $u|_{r \to \infty} < C$, where C is a bounded constant.

Because of the symmetry, it is sufficient to consider the unbounded domain, Ω ($x > 0$ and $y > 0$), and a solution u such that

$$-\Delta u + u = 0, \qquad (x, y) \in \Omega, \tag{2.1}$$

$$u = 1 \quad \text{for} \quad x = 0 \text{ and } 0 < y < 1, \tag{2.2}$$

$$\partial u/\partial x = 0 \quad \text{for} \quad x = 0 \text{ and } y > 1, \tag{2.3}$$

$$u = 1 \quad \text{for} \quad y = 0 \text{ and } x > 0. \tag{2.4}$$

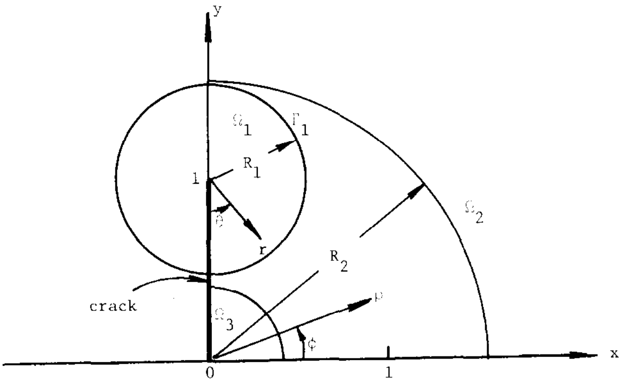

FIG. 1. The unbounded domain Ω^* along with local polar coordinates.

We notice that there exist three kinds of singularities on Ω' ($x \geqslant 0$ and $y \geqslant 0$):

1. A crack singularity at the point $(0, 1)$.
2. An "infinity" singularity, governing the solution behavior at infinity.
3. A mild singularity at the origin $(0, 0)$, near which the solutions have an asymptotic formula:

$$u = O(1) + O(\rho^2 \ln \rho) \qquad \text{as} \quad \rho \to 0.$$

Below, we try to derive solution expansions near these three singularities, one by one.

2.1. *Partial Solutions near the Crack Singularity*

Let Γ_1 be a semi-circle with center at the crack singularity $(0, 1)$, and a radius R_1 (<1), and (r, θ) be the polar coordinates shown in Fig. 1. Also denote Ω_1 as the semi-disc (r, θ) ($r < R_1$ and $0 < \theta < \pi$). Then, on Ω_1 including the crack singularity, partial solutions of Eqs. (2.1)–(2.3) also satisfy

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = u \qquad \text{for} \quad 0 < r < R_1 \text{ and } 0 < \theta < \pi, \tag{2.5}$$

$$u|_{\theta=0} = 1, \qquad \left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi} = 0. \tag{2.6}$$

Letting $u = v + \cosh(r \sin \theta)$, the problem (2.5) and (2.6) leads to a homogeneous boundary problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = v, \qquad (r, \theta) \in \Omega_1, \tag{2.7}$$

$$v|_{\theta=0} = 0, \qquad \left.\frac{\partial v}{\partial \theta}\right|_{\theta=\pi} = 0. \tag{2.8}$$

With the help of separation of variables, we can obtain

$$v = R(r)\,\Phi(\theta), \tag{2.9}$$

where the function $R(r)$ and $\Phi(\theta)$ will satisfy

$$\frac{\partial^2\Phi(\theta)}{\partial^2\theta}\Big/\Phi(\theta) = \left(r^2 - r\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)\right)\Big/R(r) = -\mu^2, \tag{2.10}$$

with a constant μ. It then follows that

$$\frac{\partial^2\Phi(\theta)}{\partial\theta^2} + \mu^2\Phi(\theta) = 0, \tag{2.11}$$

$$\Phi(0) = \left.\frac{\partial\Phi}{\partial\theta}\right|_{\theta=\pi} = 0, \tag{2.12}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) - \left(1 + \frac{\mu^2}{r^2}\right)R(r) = 0. \tag{2.13}$$

So, the solutions of (2.11) and (2.12) are

$$\sin(l + \tfrac{1}{2})\theta, \qquad l = 0, 1, \ldots \tag{2.14}$$

and the solution of (2.13) is the Bessel functions $I_\mu(r)$ for a purely imaginary argument defined by (see Tikhonov and Samarskii [10])

$$I_\mu(r) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)\,\Gamma(n+\mu+1)}\left(\frac{r}{2}\right)^{2n+\mu}. \tag{2.15}$$

Therefore, a partial solution on Ω_1 satisfying Eqs. (2.5) and (2.6) is given by

$$u = \cosh(r\sin\theta) + \sum_{l=0}^{\infty} a_n^* I_{l+1/2}(r)\sin(l+\tfrac{1}{2})\theta \qquad \text{for} \quad (r < R_1 \text{ and } 0 < \theta < \pi), \tag{2.16}$$

where a_n^* are expansion coefficients.

2.2. *Partial Solutions near Infinity*

Let (ρ, ϕ) be other polar coordinates with the origin (0, 0) shown in Fig. 1, and Ω_2 be an unbounded domain (ρ, θ) $(\rho > R_2 > 1$ and $0 < \phi < \pi/2)$. In order to find the solution in Ω_2 satisfying Eqs. (2.1), (2.3), and (2.4), we will solve the equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} = u \qquad \text{in} \quad \Omega_2, \tag{2.17}$$

with the boundary conditions

$$u|_{\phi=0}=1, \qquad \left.\frac{\partial u}{\partial \phi}\right|_{\phi=\pi/2}=0. \tag{2.18}$$

Letting $u=w+e^{-\rho\sin\phi}$, we have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 w}{\partial\phi^2}=w, \qquad (\rho,\phi)\in\Omega_2, \tag{2.19}$$

$$w|_{\phi=0}=\left.\frac{\partial w}{\partial\phi}\right|_{\phi=\pi/2}=0. \tag{2.20}$$

Similarly, by separation of variables, we obtain partial solutions near infinity

$$u=e^{-\rho\sin\phi}+\sum_{n=0}^{\infty} c_n^* K_{2n+1}(\rho)\sin(2n+1)\phi \qquad \text{for} \quad \rho>R_2 \text{ and } 0<\phi<\frac{\pi}{2}, \tag{2.21}$$

where c_n^* are real expansion coefficients, and $K_n(\rho)$ are the Hankel functions for a purely imaginary argument defined by (see [10])

$$K_n(\rho)=\frac{1}{2}\int_{-\infty}^{\infty} e^{-\rho\cosh\eta-n\eta}\,d\eta. \tag{2.22}$$

2.3. *Partial Solutions near the Mild Singularity*

In order to find partial solutions near the mild singularity (0, 0), we consider the equations:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}=u \qquad \text{for} \quad \rho<R_3<1 \text{ and } 0<\phi<\frac{\pi}{2}, \tag{2.23}$$

$$u=1 \qquad \text{for} \quad \phi=0,\pi/2. \tag{2.24}$$

Let $u=w+1$; then we have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 w}{\partial\phi^2}=(w+1) \qquad \text{for} \quad \rho<R_3 \text{ and } 0<\phi<\frac{\pi}{2}, \tag{2.25}$$

$$w=0 \qquad \text{for} \quad \phi=0,\pi/2. \tag{2.26}$$

Obviously, general solutions of Eqs. (2.25) and (2.26) can be expressed in the form

$$w=\bar{w}+\sum_{n=1}^{\infty} d_n^* I_{2n}(\rho)\sin 2n\phi, \tag{2.27}$$

where d_n^* are expansion coefficients, $\bar{w}$ is a particular solution of (2.25) and (2.26) as

$$\bar{w} = \sum_{j=1}^{\infty} w_j^*(\rho) \sin 2j\phi, \tag{2.28}$$

with the functions $w_j^*(\rho)$ that will, in detail, be provided below.

By using the Green theorem, we have, from Eq. (2.25),

$$\int_0^{R_3} \rho \, d\rho \int_0^{\pi/2} \left[\frac{\partial \bar{w}}{\partial \rho} \frac{\partial v}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial \bar{w}}{\partial \phi} \frac{\partial v}{\partial \phi} + (\bar{w} + 1) v \right] d\phi = 0. \tag{2.29}$$

Let

$$v = \psi(\rho) \sin 2j\phi, \tag{2.30}$$

$$\sum_{l=0}^{\infty} b_{2l+1} \sin 2(2l+1)\phi = 1, \tag{2.31}$$

where $\psi(\rho)$ is an arbitrarily smooth function, and the coefficients are

$$b_{2l+1} = \frac{4}{(2l+1)\pi}. \tag{2.32}$$

Then we obtain from Eqs. (2.28)–(2.32)

$$\int_0^{R_3} \left\{ -\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dw_j^*}{d\rho} \right) + \left(1 + \frac{(2j)^2}{\rho^2} \right) w_j^* + b_j \right\} \rho \psi(\rho) \, d\rho = 0, \qquad j = 2l+1, l = 0, 1, ... \tag{2.33}$$

Noting that $\psi(\rho)$ is an arbitrary function, it follows that

$$-\frac{1}{\rho} \frac{1}{d\rho} \left[\rho \frac{dw_{2l+1}^*}{d\rho} \right] + \left[1 + \frac{(2(2l+1))^2}{\rho^2} \right] w_{2l+1}^* + b_{2l+1} = 0. \tag{2.34}$$

This is the equation that governs the functions $w_{2l+1}^*(\rho)$. Besides, the functions

$$w_{2l}^*(\rho) \equiv 0, \qquad l = 0, 1, ... \tag{2.35}$$

are derived from $b_{2l} = 0$. Therefore, the solution of Eqs. (2.23) and (2.24) can be written as

$$u = 1 + \sum_{l=0}^{\infty} w_{2l+1}^*(\rho) \sin 2(2l+1)\phi + \sum_{l=1}^{\infty} d_l^* I_{2l}(\rho) \sin 2l\phi. \tag{2.36}$$

Now let us give the explicit expansions for $w^*_{2l+1}(\rho)$. According to the analysis of Lehman [4], Wigley [11], Fox and Snaker [2] and Strang and Fix [9], the function $w^*_{2l+1}(\rho)$ can be expanded into

$$w^*_j = w^*_{2l+1}(\rho) = \sum_{i=0}^{\infty} (\alpha_{j,i} + \beta_{j,i} \ln \rho)\rho^{i+2}, \tag{2.37}$$

where $j = 2l+1$, and $\alpha_{j,i}$ and $\beta_{j,i}$ are coefficients. By substituting Eq. (2.37) into Eq. (2.34), we have the identities:

$$-\sum_{i=0}^{\infty} (\alpha_{j,i}(i+2)^2 + \beta_{j,i}[(i+2)^2 \ln \rho + 2(i+2)])\rho^i$$
$$+\left[1 + \frac{(2j)^2}{\rho^2}\right] \sum_{i=0}^{\infty} (\alpha_{j,i} + \beta_{j,i} \ln \rho)\rho^{i+2} + b_j = 0, \qquad j = 2l+1,\ l = 0, 1, ... \tag{2.38}$$

Since the sum of coefficients in front of ρ_i or $\rho^i \ln \rho$ must be zero, we find that the coefficients

$$\alpha_{j,2l+1} = \beta_{j,2l+1} = 0 \qquad \text{for} \quad l = 0, 1, ... \tag{2.39}$$

and that the coefficients $\alpha_{j,2l}$ and $\beta_{j,2l}$ are defined by the following recursive formulas:

1. When $(i+2) > 2j$,

$$\beta_{j,i} = \frac{1}{(i+2)^2 - (2j)^2} \beta_{j,i-2}, \tag{2.40a}$$

$$\alpha_{j,i} = \frac{1}{(i+2)^2 - (2j)^2} [\alpha_{j,i-2} - 2(i+2)\beta_{j,i}]. \tag{2.40b}$$

2. When $(i+2) < 2j$,

$$\beta_{j,i} = 0, \tag{2.41a}$$

$$\alpha_{j,0} = \frac{b_j}{4(1-j^2)}, \qquad \alpha_{j,i} = \frac{\alpha_{j,i-2}}{(i+2)^2 - 4j^2} \qquad \text{for} \quad i \geqslant 2. \tag{2.41b}$$

3. When $i+2 = 2j$,

$$\alpha_{j,i} = 0, \tag{2.42a}$$

$$\beta_{1,0} = \frac{b_1}{4}, \qquad \beta_{j,i} = \frac{\alpha_{j,i-2}}{2(i+2)} \qquad \text{for} \quad i \geqslant 2. \tag{2.42b}$$

By now, we have already provided all the coefficients $\alpha_{j,i}$ and $\beta_{j,i}$. Consequently, we can conclude from Eqs. (2.35) and (2.36) that

$$u = 1 + \frac{1}{\pi}\rho^2 \ln \rho \qquad \text{as} \quad \rho \to 0. \tag{2.43}$$

Evidently, when $\rho \to 0$, the derivatives $\partial u/\partial \rho$ are bounded but the derivatives $\partial^2 u/\partial \rho^2 \to \infty$. This implies that the singularity at the (0, 0) is milder than the crack singularity at (0, 1) where the derivatives $\partial u/\partial r \to \infty$ as $r \to 0$ (see Eq. (2.16)). Hence we call the singularity at the origin *a mild singularity*.

3. A Boundary Approximation Method

Since there exist three kinds of singularities on Ω' ($x \geqslant 0$ and $y \geqslant 0$), we have to divide Ω into three subdomains:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3, \tag{3.1}$$

such that each of them includes only one singularity. We have found a good division shown in Fig. 2, where the subdomain Ω_i are described as follows:

1. $\Omega_1: (r < 1) \cap (y > \frac{1}{2}),$ (3.2)
2. $\Omega_3: (\rho < 1) \cap (y < \frac{1}{2}),$ (3.3)
3. Ω_2: the rest of the solution domain Ω, i.e.,

$$\Omega_2 = \Omega \backslash (\Omega_1 \cup \Omega_3). \tag{3.4}$$

In fact, we have experimented BAM with different radii r and ρ in (3.2) and (3.3) and different forms of Γ_{13}, such as a circular arc or different line segments. The condition numbers obtained by using the partition in Fig. 1 are larger than the values of *Cond* in Table I based on Fig. 2. We note that in the BAM, small condition numbers are important for obtaining extremely accurate numerical solutions.

For the partial solutions in Section 2, we choose the following piecewise expansions as admissible functions:

$$v = v_l^{(1)} = \cosh(r \sin \theta) + \sum_{l=0}^{L} \tilde{a}_l \frac{I_{l+1/2}(r)}{I_{l+1/2}(1)} \sin(l + 1/2)\theta \qquad \text{for} \quad (r, \theta) \in \Omega_1, \tag{3.5}$$

$$v = v_n^{(2)} = e^{(-\rho \sin \phi)} + \sum_{l=0}^{N} \tilde{c}_l \frac{K_{2l+1}(\rho)}{K_{2l+1}(1)} \sin(2l + 1)\phi \qquad \text{for} \quad (\rho, \phi) \in \Omega_2, \tag{3.6}$$

$$v = v_k^{(3)} = 1 + \sum_{l=0}^{N_E} w^*_{2l+1}(\rho) \sin 2(2l + 1)\phi + \sum_{l=1}^{K} \tilde{d}_l \frac{I_{2l}(\rho)}{I_{2l}(1)} \sin 2l\phi \qquad \text{for} \quad (\rho, \phi) \in \Omega_3, \tag{3.7}$$

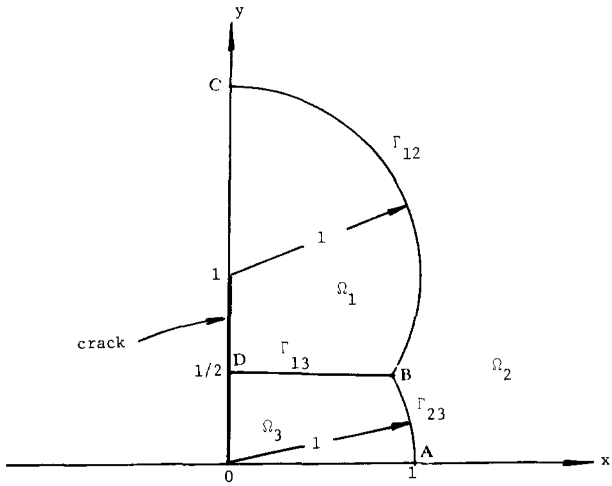

FIG. 2. A division of Ω for the BAM.

where the functions $w^*_{2l+1}(\rho)$ are defined by (2.37), $\tilde{a}_l$, $\tilde{c}_l$, and $\tilde{d}_l$ are the unknown coefficients to be found, and the integer $N_E \gg 1$. In (3.5)–(3.7), the functions $I_{l+1/2}(1)$ etc. serve as scale factors.

We note that the admissible functions (3.5), (3.6), or (3.7) satisfy Eq. (2.1) in Ω_1, Ω_2, or Ω_3 and all exterior boundary conditions (2.2)–(2.4). The coefficients $\tilde{a}_l$, $\tilde{c}_l$, and $\tilde{d}_l$ are obtained by satisfying in the least-squares sense the continuity conditions on the interior boundary Γ_{ij} only, i.e.,

$$v^{(i)} = v^{(j)} \qquad \text{and} \qquad \frac{\partial v^{(i)}}{\partial \nu} = \frac{\partial v^{(j)}}{\partial \nu}, \qquad (x, y) \in \Gamma_{ij},\ i < j, \tag{3.8}$$

where $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, and ν is the normal of Γ_{ij}.

TABLE I

Error Norms and Condition Numbers of the Numerical Solutions

L	N	K	N_E	$\lvert u-v\rvert_B$	Cond
2	0	1	299	8.475×10^{-2}	3.3
5	2	3	299	3.873×10^{-3}	13.3
8	4	5	299	4.682×10^{-4}	34.8
11	6	7	299	7.053×10^{-5}	118.9
14	8	9	299	1.193×10^{-5}	458.0
17	10	11	299	2.166×10^{-6}	1832
20	12	13	999	4.163×10^{-7}	7504
23	14	15	999	8.378×10^{-8}	31426
25	16	17	999	1.750×10^{-8}	134795

Define a space H such that

$$H=\{v \in L_2(\Omega) \mid -\Delta v+v=0 \text{ in } \Omega_i, \text{ and } v \in H^1(\Omega_i), \text{ where } i=1,2,3\}, \tag{3.9}$$

and a norm $|v|_B$ over H such that

$$|v|_B^2=[v, v]=\sum_{\substack{i,j=1\\ i<j}}^{3} \int_{\Gamma_{ij}} \left\{(v^{(i)}-v^{(j)})^2+w^2\left(\frac{\partial v^{(i)}}{\partial \nu}-\frac{\partial v^{(j)}}{\partial \nu}\right)^2\right\} ds, \tag{3.10}$$

where w is a weight constant. Then the coefficients $\tilde{a}_l$, $\tilde{c}_l$, and $\tilde{d}_l$ are chosen to minimize the norm $|u-v|_B$, i.e.,

$$|\varepsilon|_B=|u-\tilde{v}|_B=\operatorname*{Min}_{v \in H}|u-v|_B, \tag{3.11}$$

where $\tilde{v}$ is a approximate solution (see [8]).

We note that the true solution u will have zero errors in the norm $|\cdot|_B$ and will disappear in the norm definition $|\varepsilon|_B$. Also the values of $|\varepsilon|_B$ can be computed from the procedure of BAM. A useful relation between norms has, under some conditions, been established in Li [5] and Li and Mathon [7]

$$\|\varepsilon\|_H=O(\bar{M}\,|\varepsilon|_B), \tag{3.12}$$

where $\varepsilon=u-\tilde{v}$,

$$\|v\|_H=\left(\sum_{i=1}^{3}\|v\|_{1,\Omega_i}^2\right)^{1/2}, \tag{3.13}$$

and

$$\bar{M}=\operatorname{Max}(L, 2K, 2N+1), \quad w=1/\bar{M}. \tag{3.14}$$

In (3.13), $\|v\|_{1,\Omega_i}$ are the Sobolev norms, where the subdomain Ω_2 is unbounded. Based on (3.12), error norm $\|\varepsilon\|_H$ over the entire solution domain can be obtained from the error norm $|\varepsilon|_B$ only on the interior boundary Γ_{ij}.

Equation (3.11) yields a system of linear algebraic equations:

$$B\mathbf{x}=\mathbf{b}, \tag{3.15}$$

where $\mathbf{x}$ is the unknown coefficient vector with components $\tilde{a}_l$, $\tilde{c}_l$, and $\tilde{d}_l$; $\mathbf{b}$ is a known vector; and the normal matrix B is positive definite and symmetric.

All computations reported in this paper have been carried out on an IBM 4381 using double precision. The Bessel and Hankel functions for a purely imaginary argument were evaluated by using the subroutines for Bessel and Hankel functions contributed by Argonne National Laboratory. The standard integration rules such as Simpson' rule (or the trapezoidal rule) can be used for computing the integrals in (3.10) if L, K, N are not too large (as in our case).

In practice, the total number N_0 of integration nodes (i.e., the control points) should be larger than the total number M of the unknown coefficients used in (3.5)–(3.7). In our calculation $M = 3$–60, we have chosen $N_0 = 50$–200 so that $N_0 \gg M$. Therefore, the algebraic problem (3.15) never becomes singular.

4. Numerical Results

Based on the error analysis in [5, 8], the weight constant w in the approximate solution (3.11) of the model problem (2.1)–(2.4), can be chosen as

$$w = \frac{1}{\bar{M}} = \frac{1}{\mathrm{Max}(L, 2K, 2N+1)}, \tag{4.1}$$

or as in another way (see Remark 1 below). Table I gives error norms $|v|_B$ and condition numbers *Cond* of the numerical solutions. From Table I we can deduce the following asymptotic relations:

$$|\varepsilon|_B = |u - \tilde{v}|_B = O(0.755^M), \qquad \mathrm{Cond} = O(1.21^M), \tag{4.2}$$

where M denotes the total number of the unknown coefficients $\tilde{a}_l$, $\tilde{c}_l$, and $\tilde{d}_l$:

$$M = L + N + K + 2, \tag{4.3}$$

and *Cond* is the following condition number associated with the least squares problem

$$\mathrm{Cond} = \left(\frac{\lambda_{\mathrm{Max}}(B)}{\lambda_{\mathrm{Min}}(B)} \right)^{1/2}. \tag{4.4}$$

Here $\lambda_{\mathrm{Max}}(B)$ and $\lambda_{\mathrm{Min}}(B)$ are the maximal and minimal eigenvalues of the normal matrix B, respectively. Equations (4.2) exhibit exponential rates of convergence for both the error norms and condition numbers. However, the values of *Cond* grow slowly as M increases.

When $L = 25$, $N = 16$, $K = 17$, $N_E = 999$, and $w = \frac{1}{34}$, we obtained an approximate solution with

$$|\varepsilon|_B = O(10^{-8}), \qquad \mathrm{Cond} = O(10^5) \tag{4.5}$$

by using only 60 unknown coefficients, where the notation $O(10^{-8})$ means a quantity of order 10^{-8} ($\alpha 10^{-8}$, $1 \leqslant \alpha < 10$). The approximate coefficients are listed in Table II. Since the error norm in (4.5) is very small, such a solution can be regarded as an "exact solution" of Eqs. (2.1)–(2.4) for both theoretical and practical purposes. For example, in terms of an "exact solution," error norms can be computed for other approximate procedures, such as the combined methods [5, 6].

TABLE II

The Coefficients of the Approximate Solution for $L=25$, $N=16$, $K=17$, $N_E=999$, and $w=\frac{1}{34}$

A. Coefficients a_l

ℓ	a_ℓ	ℓ	a_ℓ
0	$-.1134321797585\times10^{1}$	13	$+.3130281737391\times10^{-3}$
1	$-.2035664113452\times10^{0}$	14	$+.2391633360670\times10^{-3}$
2	$+.1802645475372\times10^{0}$	15	$+.1815954074885\times10^{-3}$
3	$+.4985166606623\times10^{-1}$	16	$+.1355865655678\times10^{-3}$
4	$+.1448626726660\times10^{-1}$	17	$+.9845758791387\times10^{-4}$
5	$+.7317028845157\times10^{-2}$	18	$+.6874549027162\times10^{-4}$
6	$+.4021067893764\times10^{-2}$	19	$+.4560095585229\times10^{-4}$
7	$+.2397388765168\times10^{-2}$	20	$+.2835875926942\times10^{-4}$
8	$+.1542368986040\times10^{-2}$	21	$+.1627821904107\times10^{-4}$
9	$+.1050063183825\times10^{-2}$	22	$+.8453706771470\times10^{-5}$
10	$+.7462738867242\times10^{-3}$	23	$+.3861170957769\times10^{-5}$
11	$+.5477558835601\times10^{-3}$	24	$+.1482617318115\times10^{-5}$
12	$+.4112992391382\times10^{-3}$	25	$+.4392926896278\times10^{-6}$

B. Coefficients c_l and d_l

ℓ	c_ℓ	ℓ	d_ℓ
0	$+.2769005233385\times10^{0}$	1	$-.4015781884664\times10^{0}$
1	$-.9066552957337\times10^{-1}$	2	$+.5890728222103\times10^{-1}$
2	$+.4561996989200\times10^{-1}$	3	$-.3532773398943\times10^{-1}$
3	$-.2828278554347\times10^{-1}$	4	$+.2354001734508\times10^{-1}$
4	$+.1964155709458\times10^{-1}$	5	$-.1702868100792\times10^{-1}$
5	$-.1462958139045\times10^{-1}$	6	$+.1301850746566\times10^{-1}$
6	$+.1140500948182\times10^{-1}$	7	$-.1033923809290\times10^{-1}$
7	$-.9131571726473\times10^{-2}$	8	$+.8397076489702\times10^{-2}$
8	$+.7346837942624\times10^{-2}$	9	$-.6830460487149\times10^{-2}$
9	$-.5765893519271\times10^{-2}$	10	$+.5400913572576\times10^{-2}$
10	$+.4249769307289\times10^{-2}$	11	$-.3994984244826\times10^{-2}$
11	$-.2816615820485\times10^{-2}$	12	$+.2647170401599\times10^{-2}$
12	$+.1601231751265\times10^{-2}$	13	$-.1500088526740\times10^{-2}$
13	$-.7398900405725\times10^{-3}$	14	$+.6896932422165\times10^{-3}$
14	$+.2587534935069\times10^{-3}$	15	$-.2398708862801\times10^{-3}$
15	$-.6080346016446\times10^{-4}$	16	$+.5609893058375\times10^{-4}$
16	$+.7228909240972\times10^{-5}$	17	$-.6651837194904\times10^{-5}$

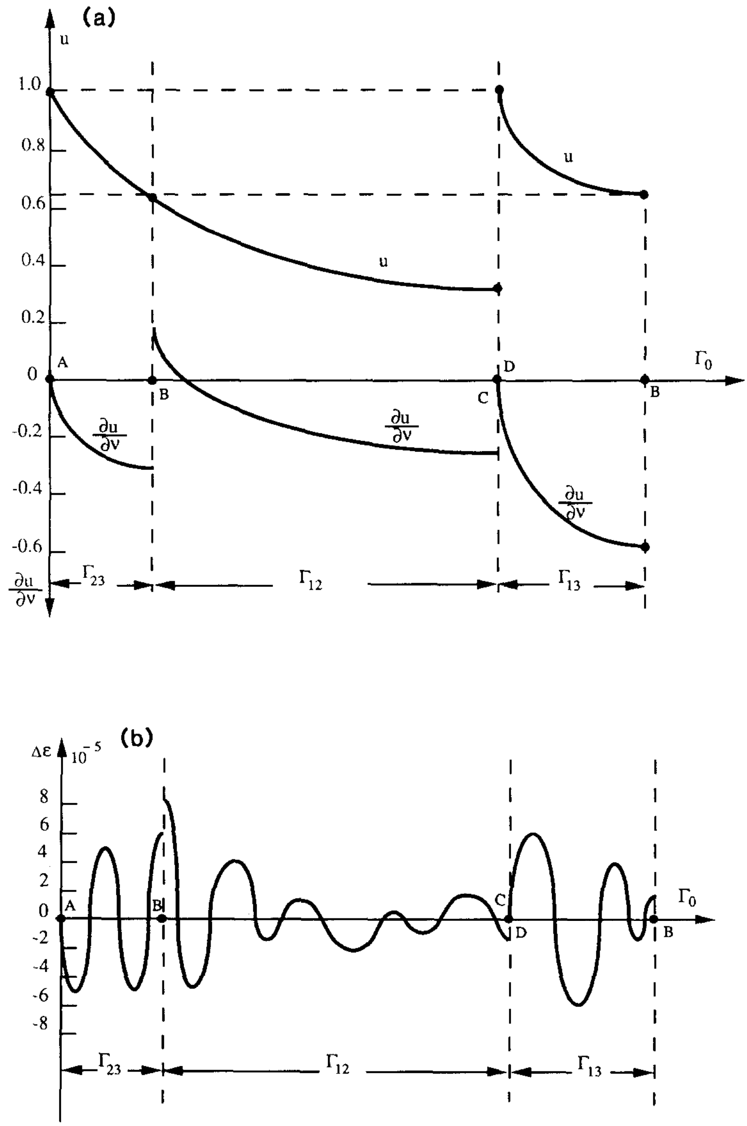

FIG. 3. (a) The graph of the approximate solution and its derivatives from the BAM with $L=11$, $N=6$, $K=7$, and $w=\frac{1}{14}$ along Γ_{23}, Γ_{12}, and Γ_{13}. (b) The graph of the errors in the approximate solution from the BAM with $L=11$, $N=6$, $K=7$, and $w=\frac{1}{14}$ along Γ_{23}, Γ_{12}, and Γ_{13}. (c) The graph of the errors in the derivatives from the BAM with $L=11$, $N=6$, $K=7$, and $w=\frac{1}{14}$ along Γ_{23}, Γ_{12}, and Γ_{13}.

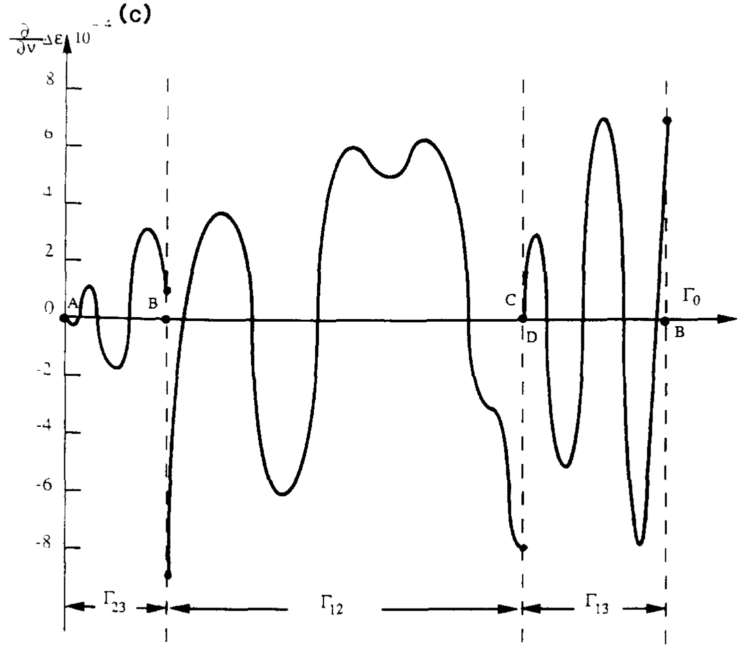

FIG. 3—*Continued.*

Since $\varepsilon = u - \tilde{v}$, the errors on the interfaces satisfy

$$\Delta\varepsilon = \varepsilon^+ - \varepsilon^- = \tilde{v}^+ - \tilde{v}^-, \qquad \text{on} \quad \Gamma_{ij},\ i<j, \tag{4.6a}$$

$$\frac{\partial}{\partial v}(\Delta\varepsilon) = \frac{\partial \varepsilon^+}{\partial v} - \frac{\partial \varepsilon^-}{\partial v} = \frac{\partial \tilde{v}^+}{\partial v} - \frac{\partial \tilde{v}^-}{\partial v}, \qquad \text{on} \quad \Gamma_{ij},\ i<j. \tag{4.6b}$$

To demonstrate the behavior of the approximate solution and its normal derivative along the interfaces we have chosen a relatively small expansion with $L=11$, $N=6$, $K=7$, and the weight constant $w=\frac{1}{14}$ from (4.1). We have used the trapezoidal rule with a total of $N_0=70$ ($>M=26$) integration nodes, yielding an error $|\varepsilon|_B = 7.053\times 10^{-5}$ and Cond $=118.9$ (see Table I). It is interesting to note that the first few coefficients in the expansion:

$$a_0 = -1.1343283038332, \qquad c_0 = 0.2768544333536, \qquad d_1 = -0.4015700184588,$$

are in good agreement with the corresponding coefficients of much longer expansion in Table II.

In Figs. 3 we display the approximate solution, its normal derivative and their errors along the interfaces. Here the horizontal axis denotes the arc length along Γ_{23}, Γ_{12}, and Γ_{13}. The values of $\Delta\varepsilon$ and $(\partial/\partial v)\,\Delta\varepsilon$ are very small, compared with the values of the approximate solution and its normal derivative (see Fig. 3a). The maximal absolute value of $(\partial/\partial \nu)\,\Delta\varepsilon$ (about 8×10^{-4}) is larger than that of $\Delta\varepsilon$ (about 8×10^{-5}) by a factor of 10. The oscillatory behavior of $\Delta\varepsilon$ and $(\partial/\partial\nu)\,\Delta\varepsilon$ along the interfaces is caused by the least-squares procedure (3.11).

To close this paper, let us make a few remarks.

1. It is worthwhile to justify our choice (4.1) of the weight constant w. Evidently, we have

$$\frac{d}{dx}x^n = nx^{n-1}, \qquad \frac{d}{d\theta}\sin n\theta = n\cos n\theta, \tag{4.7a}$$

$$\frac{d}{dr}I_\mu(r) = \frac{\mu}{r}I_\mu(r) + I_{\mu+1}(r), \qquad \frac{d}{dr}K_\mu(r) = -\frac{\mu}{r}K_\mu(r) - K_{\mu-1}(r). \tag{4.7b}$$

Then the error magnitude of the derivatives will be, in general, larger than that of the solutions (3.5)–(3.7), by a factor of $L+\frac{1}{2}$, $2K$, or $2N+1$. Therefore, we choose (4.1) as the weight constant to balance the errors in the solutions and their derivatives in the least squares procedure (3.11), and to yield the minimal error norms defined as (3.10) or (3.13) (see [5, 7, 8]).

With the choice (4.1) of the weight constant, the term

$$\frac{1}{\bar{M}^2}\int_{\Gamma_{ij}}\left(\frac{\partial v^{(i)}}{\partial \nu} - \frac{\partial v^{(j)}}{\partial \nu}\right)^2 ds \tag{4.8}$$

in (3.10) will influence the matrix B in (3.15) because the derivatives, $\partial v^{(i)}/\partial \nu$, increase by a factor of $l+\frac{1}{2}$, $2k$ or $2n+1$ (see (4.7)).

Define the norm of only the errors or derivative errors along the interface $\Gamma_0 = \bigcup_{i<j}\Gamma_{ij}$:

$$|v^+ - v^-|_{\Gamma_0} = \left\{\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{\Gamma_{ij}}(v^+ - v^-)^2\,ds\right\}^{1/2} \tag{4.9a}$$

or

$$\left|\frac{\partial v^+}{\partial \nu} - \frac{\partial v^-}{\partial \nu}\right|_{\Gamma_0} = \left\{\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{\Gamma_{ij}}\left(\frac{\partial v^{(i)}}{\partial \nu} - \frac{\partial v^{(j)}}{\partial \nu}\right)^2 ds\right\}^{1/2}, \tag{4.9b}$$

where v^+ and v^- are the values of v on two sides of Γ_0. Then we can see from (3.10) that:

$$|v^+ - v^-|_{\Gamma_0} \leqslant |\varepsilon|_B, \tag{4.10a}$$

$$\left|\frac{\partial v^+}{\partial \nu} - \frac{\partial v^-}{\partial \nu}\right|_{\Gamma_0} \leqslant \frac{1}{w}|\varepsilon|_B. \tag{4.10b}$$

For the numerical solution with $L=25$, $N=16$, $K=17$, $w=\frac{1}{34}$, we obtain from Table I,

$$|v^{+}-v^{-}|_{\Gamma_0}\leqslant 1.75\times 10^{-8}, \tag{4.11a}$$

$$\left|\frac{\partial v^{+}}{\partial v}-\frac{\partial v^{-}}{\partial v}\right|_{\Gamma_0}\leqslant 5.95\times 10^{-7}. \tag{4.11b}$$

The derivative errors on Γ_0 are of the order $O(10^{-7})$. Also the error norms $\|\varepsilon\|_H$ in the solution domain have the same order $O(10^{-7})$ from (3.12). They are very small if we recall that the derivative errors in the finite element method (FEM) using piecewise linear or quadratic interpolation functions for sufficiently smooth solutions [9], are of the order $O(h)$ or $O(h^2)$, respectively. Here h is the maximal length of triangular elements.

Besides the choice (4.1) of w, the following choice for w is also interesting in practical application. Suppose that the two error norms $|\Delta\varepsilon|_{\Gamma_0}$ and $|(\partial/\partial v)(\Delta\varepsilon)|_{\Gamma_0}$ as (4.9) are separately contemplated. Therefore, the choice of w should be connected with the expected relations of the two error norms, $|(\partial/\partial v)\,\Delta\varepsilon|_{\Gamma_0}$ and $|\Delta\varepsilon|_{\Gamma_0}$. In fact, we can expect:

$$\left|\frac{\partial}{\partial v}(\Delta\varepsilon)\right|_{\Gamma_0}\sim|\Delta\varepsilon|_{\Gamma_0},\qquad \text{if}\quad w=1, \tag{4.12a}$$

$$\left|\frac{\partial}{\partial v}(\Delta\varepsilon)\right|_{\Gamma_0}>|\Delta\varepsilon|_{\Gamma_0},\qquad \text{if}\quad 0<w\ll 1, \tag{4.12b}$$

$$\left|\frac{\partial}{\partial v}(\Delta\varepsilon)\right|_{\Gamma_0}<|\Delta\varepsilon|_{\Gamma_0},\qquad \text{if}\quad w\gg 1, \tag{4.12c}$$

compared with the choice (4.1), which causes generally slower convergence of $|(\partial/\partial v)(\Delta\varepsilon)|_{\Gamma_0}$ than that of $|\Delta\varepsilon|_{\Gamma_0}$. Evidently, this arbitrary choice of w is beneficial to stress calculation in elasticity mechanics, where the accuracy in the derivatives of displacements should be emphasized more than that in displacements themselves. In this case, we may choose $w=O(1)$, and even $w\gg 1$.

2. In Section 2, we have described techniques for finding partial solutions of elliptic equations with constant coefficients. Partial solutions of the more general equation

$$\Delta u-g(r,\theta)u=0 \tag{4.13}$$

are presented in Fox and Snaker [2] for various functions $g(r,\theta)$. Since Eq. (2.1) is a special case of (4.13), the functions from [2] can be also used to solve (2.1). This might be of interest in some engineering applications where approximate solutions of modest accuracy are required. Our choice of $I_{2l}(\rho)\sin 2l\phi$, etc. in the expansions has a wider applicability, makes the error analysis easier and leads to better and more stable numerical solutions.

3. The conditioning of the numerical problems depends on the geometric form of subdomains into which the whole region is partitioned. For example, if the subdomain Ω_i lies between two circular sectors $\bar{\Omega}_i$ and Ω_i, $\bar{\Omega}_i \supseteq \Omega_i \supseteq \Omega_i$ of radii R_i and r_i, respectively,

$$\begin{aligned} \bar{\Omega}_i &: \{(r,\theta) \mid 0 \leqslant r < R_i, 0 \leqslant \theta \leqslant \Theta\}, \\ \Omega_i &: \{(r,\theta) \mid 0 \leqslant r < r_i, 0 \leqslant \theta \leqslant \Theta\}, \end{aligned} \tag{4.14}$$

and the partial solutions $I_\mu(r) \sin \mu\theta$ are used, then the condition number of the associated coefficient matrix behaves like (see [5, 7])

$$\mathrm{Cond} = O\left(\operatorname*{Max}_i n_i \left\{\frac{R_i}{r_i}\right\}^{n_i}\right), \qquad r_i \leqslant R_i, \tag{4.15}$$

where n_i is the number of expansion terms in Ω_i. Applying (4.14) to Ω_1 and Ω_3 in Fig. 2, a horizontal section makes a good interface Γ_{13}, since the maximum ratio is closer to one. This conclusion has been also confirmed in our numerical experiments.

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